Exact representation of crossover of transitions from first order to second order in the Potts model for rumor transmission

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The L-state Potts model for rumor is the N-spin chain describing how a simple rumor transmitted by N recreant rumormongers is aggrandized. The studied rumor is represented mathematically by a simple proposition with the universal quantifier, which again is represented geometrically by a point in a proposition space. During the transmission, such a proposition is changed with the change of the rumor, which has individual number N_0 at the beginning of the transmission. Correspondingly, the point expressing the proposition may start from an arbitrary site at the proposition space, and then it shifts in the space. Thus, a spin sum of the Potts model corresponds to a walk of a point in the proposition space and spin configuration's numbers is given by enumerating the corresponding walks. The concept of the lattice path in combinatorial mathematics is introduced and the exact series representation of the configuration's numbers is given. The partition function exhibits the transition of the chain and critical equivalent inverse temperature β_c is determined. Moreover, there is a crossover value of the individual number, N_{00}. The model has a first-order transition when $N_0 < N_{00}$ and a second-order one when $N_0 > N_{00}$.

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I. INTRODUCTION

For a long time, many interesting physical rules have been found by using spin models, though they are usually very simplistic [1]. Nowadays, the experience accumulated over a long time in studying spin models is transferred to probe into various complexity phenomena in broader fields including biology, economics, sociology, etc. [2,3]. One of these attempts is the Potts chain for rumor [4], by which we describe how a simple rumor is aggrandized when it is transfered successively by a group of recreant rumormongers.

In all studies of spin models, finding exact solutions is still one of the main goals. Though it is very hard, exact solutions of a few integrable models of spins were still given, in which the celebrated one is the Onsager's solution of the two-dimensional Ising model [5]. The various integrable forms of Potts models have been studied for a long time [6–12]. At present, part of the interest in the study is concentrated on the exact solutions of nonintegrable models [13–16]. The Potts chain for rumor is one of nonintegrable statistical models because the addition for its spins does not obey the associative law. The approximate series representation of the spin-configuration number was given for this model by us [4], but the exact series representation will be given in this paper. Crossover is a complex behavior relevant to the transition appearing in some spin models [17]; it is affirmed usually through some calculation by approximate methods or through the rigorous argumentation by the renormalization group. Obviously, it is impossible to carry out the grain coarse, the key step in the renormalization transformation, in any nonintegrable system. So, we have to recur to the exact solution for affirming the crossover in the Potts model for rumor.

Rumor is the collective behavior in a society [18,19]. The primary character of all rumors is their indeterminacy. A rumor can spread along arbitrary simple or complex networks, and it is changed by rumormongers. Most studies about rumor were collected to analyze how a rumor is transferred along distinct channels [20,21]. However, the dramatic characteristic of rumor during transmission is its incessant change in semantics. Our model studies just the semantic change of the simplest rumor transmitted by a group of rumormongers along a channel without any bifurcation. Thus, each spin in this model is used to represent how a recreant rumormonger transmits and changes a rumor, but not the transmitted rumor itself. In semantics, each simplest rumor possesses at least: three semantic components the described subject, the action of the subject, and the individual number involved in the subject. For quantitatively describing a rumor, the mathematical logic [22] was used in the Ref. [4]. A rumor is abstracted as a simple proposition with the universal quantifier that $P_{m,l}=(x)_m F_l(x)$ according to the mathematic logic [4,22]. In our model, there are L predicates describing different actions and $F_l(x)$ is the lth predicate in the group. A rumor is changed during transmission, so the proposition expressing the rumor is different when a different rumormonger receives it. Therefore, number l and m of the proposition can be changed during the rumor's transmission, which express the semantic change of the transmitted rumor. Moreover, each recreant rumormonger is supposed to be able to change individual number m with only one unit, by adding one or subtracting one, according to his or her own opinion.

This paper consists of four sections and one appendix. In Sec. II, the main conception of the Potts chain for rumor and the concerned mathematical laws are reviewed simply. In Sec. III, the case of $N_0=1$ will be studied. N_0 is the individual number of an initial rumor. The configuration number is written as the power series of L according to the computational methods of lattice paths in the combinatorics [23,24]. The partition function becomes explicit as a series while configuration number is determined exactly. The system has a first-order transition at critical inverse temperature

$\beta_c = \ln(L-1)$. Correspondingly, the rumor will not be aggrandized when $\beta < \beta_c$ or be aggrandized certainly when $\beta > \beta_c$. In Sec. IV, the partition function is calculated for $N_0 > 1$. Crossover point $N_{00} = N(L-2)/L$ is determined exactly for a rumor channel of N rumormongers. The result shows that the system has a nonzero transition latent heat when $N_0 < N_{00}$, but the latent heat becomes zero when $N_0 > N_{00}$. Namely, there is the crossover phenomenon from the first-order transition to the second-order one when individual number N_0 strides over crossover point N_{00}. In the Appendix, the integer plane and various lattice paths on the plane are defined according to the combinatorics. The one to one correspondence between a lattice path and a walk on the proposition space is determined. Thus, the configuration number is given by enumerating corresponding lattice paths through the mother function (generating function).

II. SIMPLE REVIEW OF THE MODEL

A. Model

In our model, the studied rumor is a simple proposition with a universal quantifier that $P_{m,l} = (x)_m F_l(x)$ [4]. There is a group of L predicates in the model called predicate group $\{L\}$, in which the lth predicate is $F_l(x)$. Individual number m of the transmitted proposition is always a non-negative integer. The change of a rumor is expressed by changing number m and predicate $F_l(x)$. Each rumormonger has his own claim about the concerned event, which is represented by some predicate $F_l(x)$ in group $\{L\}$. Composing L branches (semiaxes) to form a skeleton, we get the proposition space (Fig. 2 in Ref. [4]). Each branch of the space is a semiaxis of the skeleton which has unit vector e_l $(1 \leqslant l \leqslant L)$. Thus, each proposition $P_{m,l}$ can be represented by positive integer m on the lth semiaxis of the skeleton as $P_{m,l} = m e_l$. All rumormongers on a rumor channel are assumed as recreant ones, who only add one to the individual number of the transmitted proposition according to their own opinions, or subtract one from the individual number.

Such a rumor channel is proved to be equivalent to a L-state spin chain $(S_1, \ldots, S_q, \ldots S_N)$ such that each spin S_q corresponds to one rumormonger. The L components of each spin are unit vectors e_l of semiaxes in the proposition space, respectively. The law of the spin's addition was shown in Ref. [4]. Obviously, the addition has neither the associative law nor the commutative law, which has never appeared in any one of fundamental physical laws. Rumor's transmission is affected by the global properties of a society such as the social guide and social acceptability degree. In our model, acceptability exponent σ of a proposition and guide exponent γ are introduced and probability of sequence $(S_1, \ldots S_q, \ldots S_N)$ is

$$p(S_1, \ldots S_q, \ldots S_N) = Q^{-1} \exp\left\{ -\left[1 - \delta\left(\sum_{q=1}^{N} S_q, r \right)(\gamma+1) \right] M/\sigma \right\}. \tag{2.1.1}$$

Moreover, we have also defined

$$\delta\left(\sum_{q=1}^{N} S_q, r \right) = \begin{cases} 1 & \text{for } k = r \\ 0 & \text{for } k \neq r, \end{cases} \quad \sum_{q=1}^{N} S_q = M e_k. \tag{2.1.2}$$

Spin sum $M = |\Sigma_{q=1}^{N} S_q|$ determines individual numbers of the proposition received by the last rumormonger, whose average value is supposed as $\bar{M}$. Ratio $\bar{M}/N$ shows how the rumor is aggrandized, which reflects the global behavior of the society.

B. Partition function

As in common, the partition function of this model can be written as

$$Q = \sum_{M=0}^{N} B_M [(L-1)e^{-M/\sigma} + e^{\gamma M/\sigma}], \tag{2.2.1}$$

in which B_M is the configuration number corresponding to spin sum $\Sigma_{q=1}^{N} S_q = M e_r$. Equation (2.2.1) is not the explicit formula because the unknown B_M is pending in it. To get the explicit series representation of the partition function, we have to count out B_M exactly. Coefficient $L-1$ of factor $e^{-M/\sigma}$ is introduced because $L-1$ predicates in predicate group $\{L\}$ are different from the one asserted by social guide $F_r(x)$.

In addition, it is worth mentioning that M will be odd if N is odd, otherwise, it will be even if N is even. Not missing its universality, we can define $N = 2n$. It will certainly make M even, which can be replaced by $2m$ $(m = 1, 2, \ldots, n)$. Accordingly, factor $e^{-M/\sigma}$ in Eq. (2.2.1) can be replaced by $e^{-2m/\sigma}$.

C. Addition of spins and walks on proposition space

The key to getting the explicit expression for partition function Q is counting the corresponding configuration numbers B_m according to spin sums $\Sigma_{q=1}^{N} S_q$. Because the addition of spins has neither the associative law nor the commutative law, we have to avoid the direct algebra calculation and appeal to the geometrical method to get the sum of N spins. If the sum of n spins is $m e_l$, the sum of $n+1$ spins may be $(m+1)e_l$ or $(m-1)e_l$, according to the addition law of spins. Sum $m e_l$ is a point (proposition) in the proposition space. Similarly, $(m+1)e_l$ and $(m-1)e_l$ both are points (propositions) in the space. The addition of spins prescribes the shift of a point on the proposition space with a unit step and the addition of N spins corresponds to an N-step walk in the proposition space.

Number m also expresses the individual number of the transmitted proposition. When the individual number of the initial rumor is $N_0 = 1$, the first step in the walk, corresponding to the spin sum, starts from original point O, and arrives at position $M e_r$ at last. But, for case $N_0 \neq 1$, the first step of the walk starts from point $N_0 e_l$, and also arrives at position $M e_r$ at last.

The translation probability of $n e_l$ shifting to $(n+1)e_l$ is written as $P(n, n+1)$ and the translation probability from $(n+1)e_l$ to $n e_l$ is $P(n+1, n)$. For a specified number, l, e_l is the unit vector of one of the L semiaxes, and all $e_k(k\neq l)$ are unit vectors of the rest of the $L-1$ semiaxes. The addition law determines that adding e_l to ne_l removes ne_l to point $(n+1)e_l$, but adding any one of the rest of $L-1$ e_k removes ne_l to point $(n-1)e_l$. So, the random walk on the proposition space corresponding to the addition of spins in a chain has the following translation probability:

$$P(n,n+1)=1/L \quad \text{for} \quad n\geqslant 0,$$

$$P(n,n-1)=(L-1)/L \quad \text{for} \quad n\geqslant 1. \qquad (2.3.1)$$

III. PARTITION FUNCTION AND FIRST-ORDER TRANSITION FOR $N_0=1$

A. Series expression of partition function

Probability sum and configuration number: If we take $\beta=\gamma/\sigma$ as the equivalent temperature, the probability sum of the model, the partition function, is

$$Q=\sum_{m=0}^{n} B_m[(L-1)e^{-2\beta m/\gamma}+e^{2\beta m}], \qquad (3.1.1)$$

where B_m is the configuration number corresponding to spin sum $\Sigma_{q=1}^{2n}S_q=2me_r$, and the social guide prefers rth predicate.

B. Spin sum and random walk

It is shown in Sec. II that adding a spin to a spin sum successively corresponds to a proposition's walk in the proposition space with translation probability (2.3.1). For the spin sum studied in the present problem, the walk starts off from the original point and moves $2n$ steps to $2me_r$ in the proposition space with L semiaxes.

In $2n$ steps of a walk, there are $n+m$ steps along the direction of semiaxes [each step is the shift from ke_l to $(k+1)e_l$] and $n-m$ steps along the opposite direction [the shift from $(k+1)e_l$ to ke_l]. The probability that a step along the direction opposite to semiaxes appears is $L-1$ times the probability that a step appears along the direction of semiaxes according to translation probability (2.3.1). Therefore, factor $(L-1)^{n-m}$ will appear in configuration number B_m, and number B_m should be written as $A_m(L-1)^{n-m}$. Moreover, a $2n$-step walk may return to the original point over and over again in the midway, and then leave the original point anew. It can return to the original point at the most $n-m$ times. Each step, starting off from the original point, can walk arbitrarily along one of the L semiaxes. Let f_k be a positive integer, then there are f_k walks returning to the original point midway k times in the total B_m walks. Then, number A_m is the sum of all numbers f_k and each f_k contains factor L^k. Lastly, configuration number B_m can be written as follows:

$$B_m=A_m(L-1)^{n-m}=\sum_{k=0}^{n-m} f_k(L-1)^{n-m}$$

$$=\sum_{k=0}^{n-m} A_{m,k}L^k(L-1)^{n-m}. \qquad (3.2.1)$$

In the right side of the above equation, all factors relevant to L have been separated off and all the remaining factors $A_{m,k}$ are independent of number L. So, we can adopt the most convenient value of L for counting out factor $A_{m,k}$, which is $L=2$. The above characteristic of configuration number B_m provides the simplest counting scheme for us as follows.

For $L=2$, Eq. (3.2.1) is simplified as

$$B_m=A_m=\sum_{k=0}^{n-m} f_k=\sum_{k=0}^{n-m} A_{m,k}2^k. \qquad (3.2.2)$$

Namely, configuration number B_m can be expressed as the power series of 2, in which $A_{m,k}$ is the coefficient of kth power of 2. Therefore, it is quite convenient that we count configuration number B_m for $L=2$ first, and then write it as the power series of 2. Number $A_{m,k}$ is given thereby as coefficients of the power series. Obviously, configuration number B_m for an arbitrary L is given by the counted $A_{m,k}$ according to Eq. (3.2.1).

C. Series expression of partition function

Random walk and lattice path. The proposition space for $L=2$ has two branches, each of which is a real semiaxis, the positive semiaxis and the negative semiaxis. So, such a space is just a real axis. Let e_1 be the unit vector of the positive semiaxis and e_2 be that of the negative axis. In $2n$ steps of a walk shifting from original point 0 to $2me_1$, there are certainly $n+m$ steps shifting along the direction of e_1 and $n-m$ steps shifts along the direction of e_2. A simple calculation shows that number A_m is just a combination number $C_{2n}^{n-m}=(2n)!/[(n-m)!(n+m)!]$.

It is very difficult to expand combination number C_{2n}^{n-m} straight away as the power series of 2, for counting factor $A_{m,k}$. So, we will adopt a convenient way to count number A_m by enumerating lattice paths according to the method given in the Appendix. An integer plane is the plane on which the Cartesian coordinate is established. Neighbor points with integer coordinates on integer plane can be linked by a line segment, and a way linked by rightward and upward line segments is called a lattice path. The equivalence between a walk on the real axis and a lattice path on the integer plane has been proved in the Appendix. In terms of this method, each walk shifting from original point 0 to $2me_1$ is just a lattice path from (0,0) to $(n+m,n-m)$ on the integer plane and A_m must be the total number of such lattice paths.

In an integer plane, the points with integer coordinates on straight line $y=x$ correspond to the original point of the real axis. Each lattice path from (0,0) to $(n+m,n-m)$ may cross straight line $y=x$ several times (as in Fig. 1). This lattice path corresponds to the 2n-step walk on the real axis from

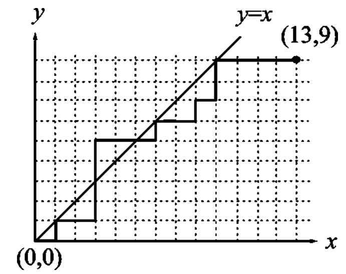

FIG. 1. The Cartesean plane and a path from (0,0) to (13,9) on it. One step on the plane increasing in x axis corresponds to one step forward in proposition space, and one step increasing in y axis corresponds to one step backward. This path on the plane can be represented by a sequence $xyxxyyyyxxxyxxyxyyxxxx$. The path has five vertices (not including the original point) on line $y=x$.

the original point to $2m$. If f_k is the number of lattice paths crossing the straight line k times midway, the sum of f_k gives A_m and $A_{m,k}$ according to the following equation

$$A_m=\sum_{k=0}^{n-m} f_k=\sum_{k=0}^{n-m} A_{m,k}L^k. \tag{3.3.1}$$

Decomposing lattice path. The conceptions of one-respect lattice path and one-direction lattice path are defined in the Appendix. Each lattice path crossing straight line $y=x$ k times (except for the starting point) can be decomposed as two segments linked end to end, the first of which is k one-respect lattice paths and the second is a one-direction lattice path. The combinatorics provides the method for enumerating lattice paths through the mother function. The mother function of the above lattice paths is the product of mother functions of the two segments of lattice paths linked. Through the mother function, we can count f_k expediently.

Mother function of the first section of lattice paths. The first section of each lattice path is k one-respect lattice paths, whose mother function is the kth power of mother function of one-respect lattice paths. The mother function of one-respect lattice paths is given in the Appendix as $1-\sqrt{1-4xy}$. So, the mother function of k one-respect lattice paths is $(1-\sqrt{1-4xy})^k$.

Mother function of the second section of lattice paths. The second section of each lattice path is a one-direction lattice path. Let $d(x,y)$ be the mother function. Each lattice path starting from original point (0,0) on the integer plane is formed by linking several one-respect lattice paths and a one-direction lattice path. All lattice paths starting from the original point on the integer plane form set S^*, whose mother function is $(1-x-y)^{-1}$, as shown in the Appendix. All lattice paths formed by linking several one-respect lattice paths belong to subset T^* of S^*, whose mother function is given in the Appendix as $(1-4xy)^{-1/2}$.

The mother function of number of lattice paths in S^* is also the product of $(1-4xy)^{-1/2}$ and $d(x,y)$. So, we have

$$d(x,y)=\frac{\sqrt{1-4xy}}{1-x-y}. \tag{3.3.2}$$

The mother function of the lattice paths crossing line $y=x$ k times is given as

$$(1-\sqrt{1-4xy})^k\frac{\sqrt{1-4xy}}{1-x-y}. \tag{3.3.3}$$

Determining A_m. Using equations

$$\frac{(1-\sqrt{1-4xy})^k}{\sqrt{1-4xy}}=\sum_{s=0}^{\infty} C_{2s+k}^{s}2^k(xy)^{s+k} \tag{3.3.4}$$

and

$$\frac{1-4xy}{1-x-y}=\sum_{p,q=0}^{\infty}(C_{p+q}^{p}-4C_{p+q-2}^{p-1})x^py^q, \tag{3.3.5}$$

we can expand Eq. (3.3.3) as the power series of 2 as follows:

$$\sum_{p,q=0}^{\infty}\sum_{s=0}^{\infty}(C_{p+q-2s-2k}^{p-s-k}-4C_{p+q-2s-2k-2}^{p-s-k-1})\times C_{2s+k}^{s}2^k(xy)^{s+k}x^{p-s-k}y^{q-s-k}, \tag{3.3.6}$$

where C_n^m is defined to be zero if $m<0$. So, the number of lattice paths from (0,0) to $(n+m,n-m)$, which cross straight line $y=x$ k times (except for the starting point), is the coefficient of term $x^{n+m}y^{n-m}$ as follows:

$$f_k=\sum_{s=0}^{n-m-k}(C_{2n-2s-2k}^{n-m-s-k}-4C_{2n-2s-2k-2}^{n-m-s-k-1})C_{2s+k}^{s}2^k. \tag{3.3.7}$$

Inserting the above equation into Eq. (3.2.1), taking index transformation $k\to k$, and $s+k\to s$, we get the series expression of A_m as

$$A_m=\sum_{s=0}^{n-m}\sum_{k=0}^{s}(C_{2n-2s}^{n-m-s}-4C_{2n-2s-2}^{n-m-s-1})C_{2s-k}^{s-k}2^k,$$

$$\text{for } (L=2). \tag{3.3.8}$$

Case of $L>2$. For the case of $L>2$, number A_m is

$$A_m=\sum_{s=0}^{n-m}\sum_{k=0}^{s}(C_{2n-2s}^{n-m-s}-4C_{2n-2s-2}^{n-m-s-1})C_{2s-k}^{s-k}L^k. \tag{3.3.9}$$

D. Analytical representation of partition function

According to the definition in Eq. (3.1.1), we get the exact series expression of the partition function from the exact series expression of the configuration number as follows:

$$Q=\sum_{m=0}^{n}\sum_{s=0}^{n-m}\sum_{k=0}^{s}(C_{2n-2s}^{n-m-s}-4C_{2n-2s-2}^{n-m-s-1})\times C_{2s-k}^{s-k}L^{k}(L-1)^{n-m}[(L-1)e^{-2\beta m/\gamma}+e^{2\beta m}]. \tag{3.4.1}$$

We study the rumor transmission along an extremely long channel, namely, the case of $n\to\infty$, which corresponds to the thermodynamic limit for a thermodynamic system. The thermodynamic study shows that property of a system is completely determined by the maximum term in the partition function. Similarly, property of rumor transmission will be provided exactly by the asymptotic analytic representation of the partition function for $n\to\infty$. Consequently, we can use various methods of the asymptotic analysis here, involving Stirling's formula and Laplace's theorem, etc. The same technique is used to study the equivalence between canonical and micro canonical ensembles in Ref. [25]. Similarly, the term with factor $(L-1)e^{-2\beta m/\gamma}$ in Eq. (3.4.1) can be cancelled. Therefore, the following part of the partition function will completely determine the statistical property of the chain

$$Q=\sum_{m=0}^{n}\sum_{s=0}^{n-m}\sum_{k=0}^{s}(C_{2n-2s}^{n-m-s}-4C_{2n-2s-2}^{n-m-s-1})\times C_{2s-k}^{s-k}L^{k}(L-1)^{n-m}e^{2\beta m}. \tag{3.4.2}$$

Sum for k. Setting $s\to\infty$, $k/s=x$, and using Stirling's formula and Laplace's formula, we get

$$\sum_{k=0}^{s}C_{2s-k}^{s}L^{k}\approx\left(\frac{L^2}{L-1}\right)^{s}\frac{2L}{L-1}. \tag{3.4.3}$$

Sum for s. Setting $n\to\infty$, and $m/n=t$, and $s/n=x$, we obtain

$$\sum_{s=0}^{n-m}(C_{2n-2s}^{n-m-s}-4C_{2n-2s-2}^{n-m-s-1})\left(\frac{L^2}{L-1}\right)^{s}\frac{L}{L-1}\approx\frac{L}{L-1}\sqrt{\frac{n}{\pi}}\int_0^{1-t}f_s(x,t)e^{nh_s(x,t)}dx, \tag{3.4.4}$$

with

$$f_s(x,t)=\frac{\sqrt{1-x}}{\sqrt{(1-x)^2-t^2}}\frac{t^2}{(1-x)^2} \tag{3.4.5}$$

and

$$h_s(x,t)=(1-x-t)\ln\frac{1-x}{1-x-t}+(1-x+t)\ln\frac{1-x}{1-x+t}+2(1-x)\ln 2+x\ln\frac{L^2}{L-1}. \tag{3.4.6}$$

The derivative of $h_s(x,t)$ with respect to x is

$$\frac{\partial h_s(x,t)}{\partial x}=\ln\frac{L^2}{4(L-1)}-\ln\frac{(1-x)^2}{(1-x)^2-t^2}. \tag{3.4.7}$$

The above equation has zeros at $x_s=1\pm Lt/(L-2)$. Obviously, the first zero $x_s=1+Lt/(L-2)$ is out of region $x\in[0,1]$, so we consider only $x_s=1-Lt/(L-2)$, the second zero. The second zero can vary with parameter t, and the value of t determines where exponent $h_s(x,t)$ has the maximum.

When $0<t<(L-2)/L$, x_s is within region $[0,1-t]$ and $h_s(x,t)$ has extreme $h_s(x,t)=2\ln L-(1+t)\ln(L-1)$ at $x_s=1-Lt/(L-2)$. According to Laplace's formula, we get

$$\frac{L}{L-1}\sqrt{\frac{n}{\pi}}\int_0^{1-t}f_s(x,t)e^{nh_s(x,t)}dx\approx\frac{L^{2n}}{(L-1)^{n(1+t)}}\frac{L-2}{L-1},\quad 0<t<\frac{(L-2)}{L}. \tag{3.4.8}$$

When $(L-2)/L<t<1$, $[\partial h_s(x,t)]/\partial x$ is negative eternally. So, $h_s(x,t)$ has maximum $h_s(0,t)=\ln 4-(1-t)\ln(1-t)-(1+t)\ln(1+t)$ at end point $x=0$ of the region. The Laplace's formula can be used to finish the integral as follows:

$$\frac{L}{L-1}\sqrt{\frac{n}{\pi}}\int_0^{1-t}f_s(x,t)e^{nh_s(x,t)}dx\approx\frac{t^2}{\sqrt{n\pi(1-t^2}}\frac{L}{L-1}\left[\ln\frac{4(L-1)}{L^2(1-t^2)}\right]^{-1}\times\frac{2^{2n}}{(1-t)^{n(1-t)}(1+t)^{n(1+t)}},\quad\frac{(L-2)}{L}<t<1. \tag{3.4.9}$$

Sum for m. We insert Eqs. (3.4.8) and (3.4.9) into Eq. (3.4.2) and transform the sum for m into an integral of t for regions $0<t<(L-2)/L$ and $(L-2)/L<t<1$, respectively. The partition function is written as $Q\approx Q_1+Q_2$, where

$$Q_1=\int_0^{(L-2)/L}L^{2n}\frac{L-2}{L-1}ne^{n[2\beta t-(1+t)\ln(L-1)-t\ln(L-1)+\ln(L-1)]}dt, \tag{3.4.10}$$

and

$$Q_2=\int_{(L-2)/L}^{1} 2^{2n}\frac{L}{L-1}\sqrt{\frac{n}{\pi(1-t^2)}}t^2\left[\ln\frac{4(L-1)}{(1-t^2)L^2}\right]^{-1}e^{n[2\beta t-(1-t)\ln(1-t)-(1+t)\ln(1+t)-t\ln(L-1)+\ln(L-1)]}dt. \quad (3.4.11)$$

Two integrals will be finished with the Laplace's theorem. Within region $[0,(L-2)/L]$, we get

$$\lim_{n\to\infty}\frac{\ln Q_1}{2n}=\begin{cases}\ln L & \text{when } \beta<\ln(L-1)\\ \ln L+\dfrac{L-2}{L}[\beta-\ln(L-1)] & \text{when } \beta>\ln(L-1).\end{cases} \quad (3.4.12)$$

Within region $[(L-2)/L,1]$,

$$\lim_{n\to\infty}\frac{\ln Q_2}{2n}=\begin{cases}\ln L-\dfrac{L-2}{L}[\ln(L-1)-\beta] & \text{when } \beta<\ln(L-1)\\ -\beta+\ln(L-1)+\ln(e^{2\beta-\ln(L-1)}+1) & \text{when } \beta>\ln(L-1).\end{cases} \quad (3.4.13)$$

E. First-order transition

Using Eqs. (3.4.12) and (3.4.13), we get

$$\lim_{n\to\infty}\frac{\ln Q}{2n}=\max\left(\lim_{n\to\infty}\frac{\ln Q_1}{2n},\lim_{n\to\infty}\frac{\ln Q_2}{2n}\right)=\begin{cases}\ln L & \text{when } \beta<\ln(L-1)\\ -\beta+\ln(L-1)+\ln(e^{2\beta-\ln(L-1)}+1) & \text{when } \beta>\ln(L-1).\end{cases} \quad (3.5.1)$$

Free energy per spin $f=-\lim_{n\to\infty}\ln Q/2\beta n$ is continuous at the critical point $\beta_c=\ln(L-1)$. However, its first-order derivative, namely, the entropy per spin, is discontinuous at this critical point. Because the increase of the entropy per spin at the critical point is

$$\Delta s=s|_{\beta\to\ln(L-1)-0}-s|_{\beta\to\ln(L-1)+0}=\frac{L-2}{L}\ln(L-1), \quad (3.5.2)$$

the system has a first-order transition at critical point $\beta_c=\ln(L-1)$. The transition latent heat per spin is $l=(L-2)/L$. Average magnetization $\overline{m}$ per spin of the chain satisfies equation

$$\lim_{n\to\infty}\frac{\overline{m}}{n}=\lim_{n\to\infty}\frac{1}{n}\frac{\partial\ln Q}{\partial\beta}=\begin{cases}0 & \text{when } \beta<\ln(L-1)\\ \dfrac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1} & \text{when } \beta>\ln(L-1).\end{cases} \quad (3.5.3)$$

Above result indicates the rumor is absolutely aggrandized when $\beta>\beta_c$. Figure 2 shows $\overline{m}/n$ varies with β for $L=3$.

As a section conclusion, we have obtained the exact solution of the Potts chain for the rumor with initial individual number $N_0=1$. Acceptability exponent σ of a proposition and guide exponent γ determine the statistical property of the rumor's transmission through equivalent inverse temperature β. The chain has a first-order transition appearing at critical equivalent inverse temperature β_c when the number of spin's components is $L>2$. Average magnetization $\overline{m}$ shows how the rumor is aggrandized. For the chain with $N=2n$ spins, we can have $\overline{m}/n>0$ only when equivalent temperature $\beta>\beta_c$, which means the social guide intensively affects the rumor transmission and the rumor is absolutely aggrandized. Otherwise, we'll get $\overline{m}/n=0$ when $\beta<\beta_c$, which means the rumor has not been aggrandized. The chain has transition latent heat $l=(L-2)/L$ at critical point β_c. It shows that the exaggeration ratio of the rumor acquires a gap at β_c.

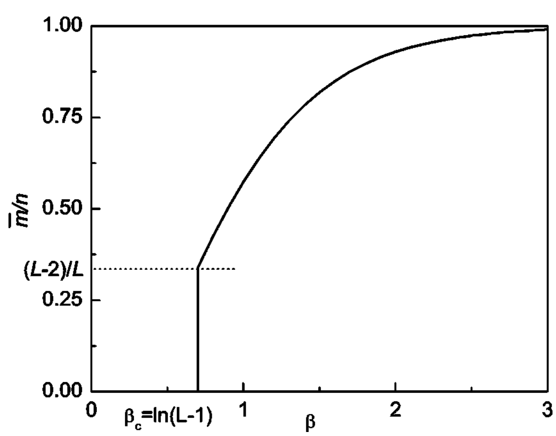

FIG. 2. $\overline{m}/n$ varies with β for $L=3$. The system has a first-order transition at critical point $\beta_c=\ln(L-1)$, which means that the rumor is absolutely aggrandized when $\beta>\beta_c$.

IV. CROSSOVER BETWEEN TRANSITIONS ALONG WITH INCREASING N_0

A. Series form of partition function

In this section, the more general form of rumor transmitted by recreant rumormongers will be studied, in which the initial proposition is $N_0 e_j$ with $N_0 \geqslant 1$. Along a rumor channel, if the individual number received by the last one in N rumormongers becomes M, the exaggeration ratio of the rumor is defined as M/N_0. Thus, the average exaggeration ratio of the rumor along various channels is $\overline{M}/N_0$. After being transmitted by N rumormongers, such a proposition becomes $N_0 e_j + \Sigma_{q=1}^{N} S_q = M e_k$. If the social guide prefers rth predicate, spin sum $N_0 e_j + \Sigma_{q=1}^{N} S_q = M e_r$ will have a larger probability than the other sum $N_0 e_j + \Sigma_{q=1}^{N} S_q = M e_k$ $(k \neq r)$.

Let $N_0 = 2n_0$, $N = 2n$, and $M = 2(n_0 + m)$, $(m > -n_0)$ as done in the preceding section, the partition function is

$$Q = \sum_{m=-n_0}^{n-2n_0} B_m[(L-1)e^{-2\beta(n_0+m)/\gamma} + e^{2\beta(n_0+m)}], \tag{4.1.1}$$

where B_m is the configuration number corresponding to spin sum $2n_0 e_j + \Sigma_{q=1}^{2n} S_q = 2m e_r$.

B. Spin sum and random walk

According to the addition of spins, adding a spin in the spin sum successively corresponds to a point's walk on the proposition space. According to the spin sum in the present problem, the walk starts off from $2n_0 e_j$ and moves certain steps randomly to $2me_r$, and $e_j \neq e_r$. Along the same way as for case $N_0 = 1$, we count configuration number B_m as follows.

In $2n$ steps of each walk from $2n_0 e_j$ to $2(n_0 + m)e_r$ on the proposition space, there are certainly $n+m$ steps walking along the direction of semiaxes and $n-m$ steps along the opposite direction. Each step along the direction opposite to semiaxes has $L-1$ choices, so B_m surely contains factor $(L-1)^{n-m}$. At one time, during movement, a proposition may start from the original point k times $(k \leqslant n-m-2n_0)$. So, we have

$$B_m = A_m(L-1)^{n-m} = \sum_{k=0}^{n-m-2n_0} A_{m,k} L^k (L-1)^{n-m}, \tag{4.2.1}$$

where $A_{m,k}$ is a factor independent of L, which can be given by considering case $L=2$.

C. Series expression of partition function

Factor $A_{m,k}$ will be given through enumerating the random walks on the real axis, as done in the preceding section. The main difference between the present problem and the foregoing one with $N_0 = 1$ is their different starting points on real axis. Therefore, each walk in the present problem is certainly linked by two parts. One part is the walk starting from point $N_0 e_j$ to the original point of the real axis and another is the walk starting from the original point, which differs from that which appeared in the foregoing problem only in its length.

The proposition space for $L=2$ is the real axis such that e_1 is the unit vector of the positive axis and e_2 is the unit vector of the negative axis. We take $j=2$ and $r=1$ for the simplification. In each walk of $2n$ steps, there are $n+m+2n_0$ steps along the same direction as e_1 and $n-m-2n_0$ steps along the same direction as e_2 with $n \geqslant m+2n_0$. Let A_m be the total number of such walks. We have

$$A_m = C_{2n}^{n-m-2n_0} = \frac{(2n)!}{(n-m-2n_0)!(n+m+2n_0)!}. \tag{4.3.1}$$

It is easily found that each walk from $2n_0 e_j$ to $2(n_0 + m)e_r, (r \neq j)$, on the proposition space corresponds to a lattice path from point $(0,2n_0)$ to point $(2n_0+n+m, n-m)$ on the integer plane. Because the start point and end point are assumed to seat at different semiaxes (the condition), such a walk goes through the original point at least once. So, the corresponding lattice path crosses straight line $x=y$ at least once.

Let (n_1, n_1) be the first crossing of such a lattice path with straight line $x=y$. As shown above, each studied walk can be separated into two sections and the corresponding lattice path can also be separated into two sections. The first is a lattice path from $(0,2n_0)$ to (n_1, n_1) and the second is the lattice path from (n_1, n_1) to $(2n_0+n+m, n-m)$. In counting, we make the reflection transformation only for the first section with respect to straight line $x=y$, under which point $(0,2n_0)$ is transformed into point $(2n_0,0)$ and this section is transformed into a lattice path from $(2n_0,0)$ to (n_1, n_1); the whole path is transformed into a lattice path from $(2n_0,0)$ to $(2n_0+n+m, n-m)$. The reflection is a one to one transformation, so A_m also can be regarded as the total number of all lattice paths from $(2n_0,0)$ to $(2n_0+n+m, n-m)$, which crosses straight line $y=x$ midways at least once. We will adopt this form of lattice paths in counting number A_m.

Separating lattice path. Similarly, we separate such a lattice path into two sections in the following way. The first section starts from point $(2n_0,0)$ and ends at the first crossing on straight line $y=x$. The second section starts from straight line $y=x$ and ends at point $(2n_0+n+m, n-m)$, which includes k one-respect lattice paths and a one-direction lattice path. The mother function for the whole lattice paths is the product of mother functions for the two sections of the lattice paths, which will be determined, respectively.

Mother function of first section of lattice paths. An arbitrary lattice path from $(2n_0,0)$ to straight line $y=x$ can be separated into a lattice path from $(2n_0,0)$ to the first crossing on straight line $y=x$, which is just the above-mentioned first section of lattice paths, and several one-respect lattice paths. Such a lattice path can cross straight line $y=x$ midway several times. The mother function for the first section of lattice paths is the quotient of mother functions for the lattice paths from $(2n_0,0)$ to straight line $y=x$ and that of several one-respect lattice paths.

Consider the lattice paths starting from $(2n_0,0)$ and ending at point $(2n_0+k,2n_0+k)$ on straight line $y=x$. Each lattice path from $(2n_0,0)$ to $(2n_0+k,2n_0+k)$ has $2n_0+2k$ steps in which, $2n_0+k$ steps are upward and k steps are rightward. So, $C^k_{2n_0+2k}$ is the number of the lattice paths, and the mother function of these lattice paths is $\Sigma_{k=0}^{\infty}C^k_{2n_0+2k}y^kx^{2n_0+k}$ according to that given in the appendix.

Supposing $t=xy$ and using equation

$$\frac{1}{\sqrt{1-4t}}\left(\frac{1-\sqrt{1-4t}}{2t}\right)^l=\sum_{n=0}^{\infty}C^n_{2n+l}t^n, \qquad (4.3.2)$$

we obtain the mother function of the lattice paths from $(2n_0,0)$ to straight line $y=x$ as follows

$$\sum_{k=0}^{\infty}C^k_{2n_0+2k}x^ky^{2n_0+k}=\sum_{k=0}^{\infty}C^k_{2n_0+2k}t^ky^{2n_0}=\frac{1}{\sqrt{1-4t}}\left(\frac{1-\sqrt{1-4t}}{2t}\right)^{2n_0}y^{2n_0}. \qquad (4.3.3)$$

A lattice path from $(0,0)$ to straight line $y=x$ is formed by linking several one-respect lattice paths. The mother function of the number of such lattice paths is $\Sigma_{k=0}^{\infty}C^k_{2k}x^ky^k$ with

$$\sum_{k=0}^{\infty}C^k_{2k}x^ky^k=\frac{1}{\sqrt{1-4xy}}. \qquad (4.3.4)$$

The mother function of the number of the lattice paths from $(2n_0,0)$ to the first crossing on straight line $y=x$ (never crossing straight line $y=x$ midway) is supposed to be $f_1(x,y)$, which is thereby the quotient of above-counting mother functions as follows:

$$f_1(x,y)=\frac{\sum_{k=0}^{\infty}C^k_{2n_0+2k}x^ky^{2n_0+k}}{\sum_{k=0}^{\infty}C^k_{2k}x^ky^k}=\left(\frac{1-\sqrt{1-4xy}}{2x}\right)^{2n_0}. \qquad (4.3.5)$$

Mother function of second section of lattice paths. The second section is the lattice path from straight line $y=x$ to $(2n_0+n+m,n-m)$, which returns to the straight line k times midway. The mother function of the number of these lattice paths is

$$f_2(x,y)=(1-\sqrt{1-4xy})^k\frac{\sqrt{1-4xy}}{1-x-y}. \qquad (4.3.6)$$

Mother function of total lattice paths. The mother function for the number of the whole lattice paths starting from $(2n_0,0)$ and crossing straight line $y=x$, $k+1(k\geqslant 0)$ times is the product of $f_1(x,y)$ and $f_2(x,y)$ as follows:

$$g_k(x,y)=f_1(x,y)f_2(x,y)=\frac{(1-\sqrt{1-4xy})^{2n_0+k}}{(2x)^{2n_0}}\frac{\sqrt{1-4xy}}{1-x-y}. \qquad (4.3.7)$$

Determining $A_{m,k}$. Using Eqs. (3.3.4) and (3.3.5), we can write Eq. (4.3.7) as follows:

$$g_k(x,y)=\sum_{s=0}^{\infty}\sum_{p,q=0}^{\infty}(C^p_{p+q}-4C^{p-1}_{p+q-2})\times C^s_{2s+2n_0+k}2^kx^{p+s+k}y^{q+s+k+2n_0}. \qquad (4.3.8)$$

Let $n+m=p+s+k$ and $n-m=q+s+2n_0$. Then, coefficient of term $x^{n+m}y^{n-m}$ in mother function $g_k(x,y)$ is

$$f_k=\sum_{s=0}^{n-m-k-2n_0}(C^{n-m-s-k-2n_0}_{2n-2s-2k-2n_0}-4C^{n-m-s-k-2n_0-1}_{2n-2s-2k-2n_0-2})C^s_{2s+2n_0+k}2^k. \qquad (4.3.9)$$

Obviously, $f_k=A_{m,k}2^k$. So, we have

$$A_{m,k}=\sum_{s=0}^{n-m-k-2n_0}(C^{n-m-s-k-2n_0}_{2n-2s-2k-2n_0}-4C^{n-m-s-k-2n_0-1}_{2n-2s-2k-2n_0-2})C^s_{2s+2n_0+k}. \qquad (4.3.10)$$

D. Counting partition function

Inserting Eq. (4.3.10) into Eq. (4.2.1), we get the expression of the partition function as follows:

$$Q=\sum_{m=-n_0}^{n-2n_0}\sum_{s=0}^{n-m-2n_0}\sum_{k=0}^{s}(C^{n-m-s-2n_0}_{2n-2s-2n_0}-4C^{n-m-s-2n_0-1}_{2n-2s-2n_0-2})C^{s-k}_{2s-k+2n_0}L^k(L-1)^{n-m}\times[(L-1)e^{-2\beta(n_0+m)/\gamma}+e^{2\beta(n_0+m)}]. \qquad (4.4.1)$$

For $n\to\infty$, the property of the system is determined entirely by the maximum term in the series expression of the partition function. So, term $(L-1)e^{-2\beta(n_0+m)/\gamma}$ in Eq. (4.4.1) will be taken out and the partition function becomes

$$Q=\sum_{m=-n_0}^{n-2n_0}\sum_{s=0}^{n-m-2n_0}\sum_{k=0}^{s}(C^{n-m-s-2n_0}_{2n-2s-2n_0}-4C^{n-m-s-2n_0-1}_{2n-2s-2n_0-2})C^{s-k}_{2s-k+2n_0}L^k(L-1)^{n-m}e^{2\beta(n_0+m)}. \qquad (4.4.2)$$

Each summing in the equation will be changed to an integral by the Stirling's formula and the integral will be finished by using the Laplace's formula.

Sum for k. For $n,s\to\infty$, setting $s/n=y$, $n_0/n=y_0$, and $k/n=x$, we obtain

$$\sum_{k=0}^{s} C_{2s+2n_0-k}^{s-k} L^k \approx \sqrt{\frac{n}{\pi}} \int_0^y f_k(x) e^{s h_k(x)} dx, \quad (4.4.3)$$

where

$$f_k(x) = \sqrt{\left(y+y_0-\frac{x}{2}\right) \Big/ [(y+2y_0)(y-x)},$$

and

$$h_k(x) = (2y+2y_0-x)\ln(2y+2y_0-x) - (y+2y_0)\ln(y+2y_0) - (y-x)\ln(y-x) + x\ln L. \quad (4.4.4)$$

Let us count extremum point x_0 of function $h_k(x)$ for $x \in [0,y]$. $h_k(x)$ has maximum at point x_0 with

$$x_0 = \begin{cases} 0 & y < 2y_0/(L-2) \\ y-(y+2y_0)/(L-1) & y > 2y_0/(L-2). \end{cases} \quad (4.4.5)$$

According to the Laplace's theorem about integral, we obtain

$$\sum_{k=0}^{s} C_{2s+2n_0-k}^{s-k} L^k \propto e^{n h_k(x_0)}. \quad (4.4.6)$$

Over here, we are only concerned about how the partition function is increasing with increscent n for $n \to \infty$. So, we can keep down only the exponent function of $n h_1(x_0)$ in Eq. (4.4.6).

Sum for s. Let $m/n = t$. using the Stirling's formula for $n \to \infty$, we have

$$\sum_{s=0}^{n-m-2n_0} C_{2n-2n_0-2s}^{n-2n_0-m-s} e^{n h_k(x_0)} \propto \int_0^{1-2y_0-t} e^{n h_s(y)} dy, \quad (4.4.7)$$

where

$$h_s(y) = (2-2y_0-2y)\ln(2-2y_0-2y) - (1-2y_0-t-y)\ln(1-2y_0-t-y) - (1+t-y)\ln(1+t-y) + h_k(x_0).$$

We denote the point where $h_s(y)$ has the maximum by $\overline{y}$. When $y_0 > (L-2)/L$, $\overline{y} = (1-t-2y_0)/(t+2y_0y_0)$. When $y_0 < (L-2)/L$, we must discuss for different regions of t. If $t \in (-y_0, 1-2y_0-2/L)$ then $\overline{y} = 1-[Lt+2(L-1)y_0]/(L-2)$. If $t \in (1-2y_0-2/L, 1-2y_0)$ then $\overline{y} = (1-t-2y_0)/(t+2y_0y_0)$.

Sum for m. Lastly, we should change the sum for m to the integral for t and finish the following integral of Q:

$$Q \propto \int_{-y_0}^{1-2y_0} e^{n[h_s(\overline{y})+2\beta t+2\beta y_0 - t\ln(L-1)+\ln(L-1)]} dt. \quad (4.4.8)$$

Setting

$$h(t) = h_s(\overline{y}) + 2\beta t + 2\beta y_0 - t\ln(L-1) + \ln(L-1), \quad (4.4.9)$$

maximum $h(\overline{t})$ of $h(t)$ within region $t \in (-y_0, 1-2y_0)$ will be counted.

When $y_0 > (L-2)/L$,

$$h(\overline{t}) = \begin{cases} 2\ln 2 - (1-y_0)\ln(1-y_0) - (1+y_0)[\ln(1+y_0) - \ln(L-1)], & \dfrac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1} < y_0 \\ 2\ln(e^{2\beta-\ln(L-1)}+1) - 2[\beta - \ln(L-1)](1+y_0), & \dfrac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1} > y_0. \end{cases} \quad (4.4.10)$$

For $y_0 < (L-2)/L$,

$$h(\overline{t}) = \max\left[h(\overline{t_1}), h(\overline{t_2})\right] = \begin{cases} 2\ln L, & \beta < \ln(L-1) \\ 2\ln(e^{2\beta-\ln(L-1)}+1) - 2[\beta-\ln(L-1)](1+y_0), & \beta > \ln(L-1). \end{cases} \quad (4.4.11)$$

E. Second-order transition and crossover from first-order transition into second-order transition

The above result depends strictly on individual number $N_0 = 2n_0$ of the initial proposition. It is necessary to study how the global properties of the spin chain change with the change in number N_0.

When $y_0 > (L-2)/L [N_0 > N(L-2)/L]$, free energy per spin f of the system is defined according to Eqs. (4.4.8) and (4.4.10) as follows:

$$f = -\lim_{n\to\infty} \frac{\ln Q}{2\beta n} = -\frac{h(\overline{t})}{2\beta}. \quad (4.5.1)$$

The entropy per spin is

$$s=\beta^2\frac{\partial f}{\partial\beta}=\begin{cases}\ln 2-\frac{1}{2}(1-y_0)\ln(1-y_0)-\frac{1}{2}[1+y_0-\ln(L-1)]\ln(1+y_0), & \frac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1}<y_0\\ \ln(e^{2\beta-\ln(L-1)}+1)-\frac{2\beta e^{2\beta-\ln(L-1)}}{e^{2\beta-\ln(L-1)}+1}+(1+y_0)\ln(L-1), & \frac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1}>y_0.\end{cases} \tag{4.5.2}$$

It is easy to find that f and s are continuous at critical point $\beta_{c2}=\ln(L-1)/2+\frac{1}{2}\ln[(1+y_0)/(1-y_0)]$, but first-order derivative of entropy s is discontinuous at the critical point. Namely, we have

$$\frac{\partial s}{\partial\beta}=\begin{cases}0, & \beta<\beta_{c2}\\ -4\beta\frac{e^{2\beta-\ln(L-1)}}{(e^{2\beta-\ln(L-1)}+1)^2}, & \beta>\beta_{c2}.\end{cases} \tag{4.5.3}$$

Thus, the system has second-order transition at critical point $\beta=\beta_{c2}$ for case $y_0>(L-2)/L$. Average $\overline{m}$ of the spin chain satisfies the following equation

$$\lim_{n\to\infty}\frac{\overline{m}+n_0}{n}=\lim_{n\to\infty}\frac{1}{2n}\frac{\partial\ln Q}{\partial\beta}$$
$$=\begin{cases}0 & \text{when}\ \ \beta<\beta_{c2}\\ \frac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1}-y_0 & \text{when}\ \ \beta>\beta_{c2}.\end{cases} \tag{4.5.4}$$

This result means that the rumor cannot be magnified when $\beta<\beta_{c2}$. But when $\beta>\beta_{c2}$, due to inequality $(e^{2\beta-\ln(L-1)}-1)/(e^{2\beta-\ln(L-1)}+1)-y_0>0$, the rumor can be magnified. [Strictly speaking, average exaggeration ratio $\overline{M}/N_0\le 1$ when $(e^{2\beta-\ln(L-1)}-1)/(e^{2\beta-\ln(L-1)}+1)-y_0<y_0$, and $\overline{M}/N_0>1$ when $(e^{2\beta-\ln(L-1)}-1)/(e^{2\beta-\ln(L-1)}+1)-y_0>y_0$.] The transition point is determined by the acceptability exponential and the society's guide.

However, when $y_0<(L-2)/L$ $[N_0<N(L-2)/L]$, free energy per spin f is given according to Eqs. (4.4.8) and (4.4.11) as follows

$$f=-\lim_{n\to\infty}\frac{\ln Q}{2\beta n}=-\frac{1}{2\beta}\begin{cases}2\ln L, & \beta<\ln(L-1)\\ 2\ln(e^{2\beta-\ln(L-1)}+1)-2[\beta-\ln(L-1)](1+y_0), & \beta>\ln(L-1).\end{cases} \tag{4.5.5}$$

Moreover, the entropy per spin is

$$s=\beta^2\frac{\partial f}{\partial\beta}=\begin{cases}\ln L, & \beta<\ln(L-1)\\ \ln(e^{2\beta-\ln(L-1)}+1)-\frac{2\beta e^{2\beta-\ln(L-1)}}{e^{2\beta-\ln(L-1)}+1}+(1+y_0)\ln(L-1), & \beta>\ln(L-1).\end{cases} \tag{4.5.6}$$

It is easy to find that f is continuous at critical point $\beta_c=\ln(L-1)$ but entropy s is not. The increment of the entropy in the critical point is

$$\Delta s=s|_{\beta\to\ln(L-1)-0}-s|_{\beta\to\ln(L-1)+0}$$
$$=\left(1-2y_0-\frac{2}{L}\right)\ln(L-1). \tag{4.5.7}$$

Thus, the system has first-order transition at critical point $\beta=\beta_c$ and transition latent heat of per spin is $l=(1-2y_0-2/L)$. Average magnetization $\overline{m}$ per spin satisfies equation

$$\lim_{n\to\infty}\frac{\overline{m}+n_0}{n}=\lim_{n\to\infty}\frac{1}{2n}\frac{\partial\ln Q}{\partial\beta}$$
$$=\begin{cases}0 & \text{when}\ \ \beta<\ln(L-1)\\ \frac{e^{2\beta-\ln(L-1)}-1}{e^{2\beta-\ln(L-1)}+1}-y_0 & \text{when}\ \ \beta>\ln(L-1).\end{cases} \tag{4.5.8}$$

This result means that the rumor cannot be aggrandized when $\beta<\beta_c$. On the other hand, for $\beta>\beta_c$, the rumor can be aggrandized because $(e^{2\beta-\ln(L-1)}-1)/(e^{2\beta-\ln(L-1)}+1)-y_0$

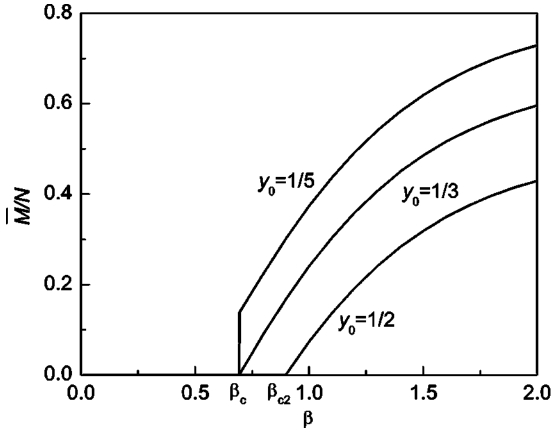

FIG. 3. $\overline{M}/N$ varies with β for $L=3$. The curves for $y_0=1/2$ and $y_0=1/5$ represent the second-order transition and the first order-transition, respectively. The curve for $y_0=1/3$ represents the crossover of transition because of $y_0=(L-2)/L$. The second-order transition happens at critical point $\beta_{c2}=[\ln(L-1)]/2+\frac{1}{2}\ln[(1+y_0)/(1-y_0)]$ for $y_0>(L-2)/L$. The first-order transition happens at critical point $\beta_c=\ln(L-1)$ for $y_0<(L-2)/L$.

>0. [Strictly speaking, average exaggeration ratio $\overline{M}/N_0<1$ when $(e^{2\beta-\ln(L-1)}-1)/(e^{2\beta-\ln(L-1)}+1)-y_0<y_0$ and $\overline{M}/N_0>1$ when $(e^{2\beta-\ln(L-1)}-1)/(e^{2\beta-\ln(L-1)}+1)-y_0>y_0$.] Figure 3 shows $\overline{M}/N$ varies with β for $L=3$. The curves for $y_0=1/2$ and $y_0=1/5$ represent the second-order transition and the first-order transition, respectively. The curve for $y_0=1/3$ represents the crossover of transition because of $y_0=(L-2)/L$.

It is easy to see that point $y_0=(L-2)/L$, namely, individual's number $N_0=[(L-2)/L]N=N_{00}$, is the crossover point at which the crossover between first-order and second-order transition in the spin chain comes true.

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APPENDIX: ENUMERATION OF THE LATTICE PATHS

For a given positive integer M, number of solutions $\{S_q\}$ determined by spin sum $\Sigma_{q=1}^{N}S_q=Me_r$, or the number of spin configurations, is the key to giving the exact solution of the rumor model.

The addition of spins shows that the spin polynomial determines a translation at a proposition space from the original point to Me_r. According to the addition law, the configuration numbers for a spin sum is just the total number of the random walks with transition probability Eq. (2.3.1).

Because the original point holds a particular place at a proposition space, which is the joint of L semiaxes, the transition probability for case $L>2$ has the central symmetry with respect to the original point, but not the translation invariance. So, the calculation of the configuration number is so complicated that we have to recourse to the concept of the lattice path in combination mathematics.

1. Lattice path

a. Basic conceptions

Integer plane. An integer plane is the plane on which the Cartesian coordinate is established. Theoretically, the integer plane involves all four quadrants of the plane, but actually we will pay attention only to the first quadrant. Neighbor points with integer coordinates are linked by segments with unit length. In order to simplify description, we will call the set of all points with integer coordinates on a plane and all segments linking such points as the integer plane later.

Lattice path. A way turning rightwards and upwards on the integer plane is called a lattice path if it is formed by linking.

Total number of lattice paths. The length of a lattice path from (0,0) to $(m,n)(m\geqslant 0,n\geqslant 0)$ is $m+n$, in which m steps turn rightwards and n steps turn upwards. Out of the $m+n$ steps, upward n steps have C_{m+n}^{n} ways to appear in the lattice path, so the total number of the lattice paths from (0,0) to (m,n) is $C_{m+n}^{n}=(m+n)!/(m!n!)$.

In counting the number of some type of the lattice paths, we must show clearly the essential characteristics of the lattice paths. Two types of the lattice paths appearing frequently are defined as follows.

One-respect lattice path. Straight line $x=y$ divides an integer plane into two parts (two respects). The lattice path starting from a point (x_i,y_i) on straight line $x=y$ and then returning to straight line $x=y$ is called a one-respect lattice path if it does not cross straight line $y=x$ midway.

One-direction lattice path. The lattice path starting from straight line $x=y$ and then arriving at (x_n,y_n) is called a one-direction lattice path if it does not involve any one-respect lattice path. Unlike a one-respect lattice path, none of the one-direction lattice path ever returns to straight line $y=x$ midway.

b. Lattice path and random walk on real axis

In practice, each lattice path on an integer plane corresponds to a walk on a straight line one to one.

Correspondence between integer plane and real axis. A straight line becomes the real axis when a coordinate is established on it. Let point (0,0) on an integer plane correspond to original point 0 of the real axis. Now we study the shift among points on the real axis. Each step starting from a point n and arriving at point $n+1$ on the real axis corresponds to a step of a lattice path turning rightward, and each step from $n+1$ to n on the real axis corresponds to an upward step of a lattice path. So, each point (x_i,y_i) on the integer plane corresponds to the point with coordinate $n=x_n-y_n$ on the real axis. Moreover, each point (x_i,y_i) over straight line $x=y$ has $x_i<y_i$ which corresponds to a point on the negative semiaxis. Similarly, each point (x_i,y_i) under straight line $x=y$ corresponds to a point on the positive semiaxis.

Lattice path and random walk. Let the starting point of a lattice path be (0,0) and the end point be (x_n,y_n). This lattice path corresponds to a walk on the real axis from the original point 0 to point $n=x_n-y_n$. So, the total number of walks on the real axis from the original point to point n

$=x_n-y_n$ equals the total number of the lattice paths from (0,0) to (x_n, y_n). Therefore, we can enumerate walks on the real axis through enumerating the corresponding lattice paths.

One-axis walk. There is a walk starting from original point 0 and going only on the negative or on the positive axis, and finally returning to 0 (does not involve the original point in the midway). We call such a walk as a one-axis walk. Obviously, such a walk corresponds to the one-respect lattice path.

One-direction walk. A walk on the positive or on the negative axis from original point 0 to a point m (does not return to the original point in the midway) is called a one-direction walk. Obviously, such a walk corresponds to the one-direction lattice path.

In order to count number $A_{m,k}$, we have to use the alternative method in enumerating lattice paths. A walk on the real axis from 0 to m may be regarded as a walk formed by linking several one-axis walks and a one-direction walk end to end. At one time, the corresponding lattice path is regarded as a path formed by linking several one-respect lattice paths and a one-direction lattice path end to end.

We consider a $2n$-step walk on the real axis from 0 to m, which is formed by linking several one-axis walks and a one-direction walk. Each one-axis walk may go on different semiaxis and have different step numbers. So, various possible one-axis walks and one-direction walk must be considered in the calculation of the total number of random walks. Similarly, various possible one-respect lattice paths and one-direction lattice paths must be considered in the calculation of the total number of lattice paths, too. The following theorems can be proved easily.

Theorem A.1.1. The number of lattice paths from (0,0) to (m,n) $(m \geqslant n \geqslant 0)$ but never surpassing straight line $y=x$ is

$$C_{n+m}^{n}-C_{n+m}^{n-1}. \tag{A.1.1}$$

Obviously, this number is relevant to that of the ways of the walk from original point 0 to point m on a semiaxis. Similarly, we have

Theorem A.1.2. The number of lattice paths from (1,0) to (m,n) $(m \geqslant n \geqslant 0)$ without surpassing straight line $y=x-1$ is

$$C_{n-1+m}^{n}-C_{n-1+m}^{n-1}. \tag{A.1.2}$$

2. Free monomial system

a. Basic conception

For the calculation of the total number of various lattice paths, relevant content of free monomials is needed. Let S be a set with several elements. For an arbitrary non-negative integer m, m elements of set S can form a sequence (elements in the sequence can be repeated). With all such sequences, we can form a new set, which is denoted as S^m. The length of every element $u \in S^m$ is m, which is written as $m=\mathrm{len}\,u$. There is only one sequence with length 0, which is the empty sequence. It can be denoted as 1. Thus, $S^0=\{1\}$ and $S^1=\{S\}$.

Monomial and free monomial system. Different number m corresponds to the sequence with different length, which belongs to different set S^m. All finite sequences composed by elements in set S form a new set, which is noted as S^*, and we have

$$S^*=\cup_{j=0}^{\infty} S^i. \tag{A.2.1}$$

S^* is regarded as being generated by S, and elements of $S^1=S$ are essential elements of S^*. Also, elements in S^* are called monomials.

The following juxtaposition operation $\circ$ can be defined as follows:

For two arbitrary elements $a_1 a_2 \cdots a_m \in S^*$ and $b_1 b_2 \cdots b_n \in S^*$, their juxtaposition is another element $a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n$ in S^*, which is denoted as follows

$$(a_1 a_2 \cdots a_m)\circ(b_1 b_2 \cdots b_n)=a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n. \tag{A.2.2}$$

Obviously, juxtaposition operation $\circ$ satisfies the associative law, and empty sequence 1 is its unique unit element. So, $(S^*,\circ)$ is a unitary semigroup. $(S^*,\circ)$ is called the free monomial system on S, or for short, the free monomial system. Elements of S may be common variables. Let $S=\{x\}$, then $S^m=\{x^m\}$. So, $S^*=\{1,x,x^2,\ldots\}$, which is right the usual monomial system (power of x). If $S=\{x,y\}$, where x and y represent turning one step rightwards and one step upwards on an integer plan, respectively, then, the element of S^m is a lattice path with length m. S^* is the set of all lattice paths starting from the original point. Juxtaposition operation $\circ$ denotes linking two lattice paths end to end.

Theorem A.2.1. For every $u \in S^*$ and $a \in S$, we have

$$\sum_{u \in S^*} u=\left(1-\sum_{a \in S} a\right)^{-1}. \tag{A.2.3}$$

Proof: $\Sigma_{u \in S^*} u=\Sigma_{n=0}^{\infty}\Sigma_{u \in S^n} u=\Sigma_{n=0}^{\infty}(\Sigma_{a \in S} a)^n=(1-\Sigma_{a \in S} a)^{-1}$.

b. Mother function for lattice paths

Mother function. Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a number series and x a variable, the mother function for numbers a_n is defined as the following power series

$$f(x)=a_0+a_1 x+a_2 x^2+\cdots+a_n x^n+\cdots=\sum_{n=0}^{\infty} a_n x^n. \tag{A.2.4}$$

Let α be a real number and k a non-negative integer, and denote $\binom{\alpha}{k}=[\alpha(\alpha-1)\ldots(\alpha-k+1)]/k!$. We denote $\binom{\alpha}{k}=C_\alpha^k$ when α is a non-negative integer. The mother function for binomial coefficients $\binom{\alpha}{n}$ is

$$f_\alpha(x)=\binom{\alpha}{0}+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\cdots+\binom{\alpha}{n}x^n+\cdots. \tag{A.2.5}$$

Because it is consistent with the expansion of $(1+x)^{\alpha}$ in the manner of the mathematical analysis, we denote $f_{\alpha}(x)$ as $(1+x)^{\alpha}$. Taking $\alpha=-1/2$, we have

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n=\sum_{n=0}^{\infty}C_{2n}^{n}x^n=(1-4x)^{-1/2}. \quad (A.2.6)$$

Mother function for S^*. Set S^* is the set of all lattice paths starting from the original point, which is generated by set $S=\{x,y\}$. The mother function for it is obtained according to Theorem A.2.1:

$$\sum_{u\in S^*}u=\left(1-\sum_{a\in S}a\right)^{-1}=(1-x-y)^{-1}. \quad (A.2.7)$$

The above formula can also be rewritten as

$$(1-x-y)^{-1}=\sum_{k=0}^{\infty}(x+y)^k=\sum_{m,n}l_{m.n}x^my^n, \quad (A.2.8)$$

where $l_{m,n}$ is the total number of lattice paths from (0,0) to (m,n).

c. Subsystem of monomials

Subsystem about one-respect lattice path T^*, T and corresponding mother function. Set T^* of all lattice paths starting from straight line $y=x$ then returning to it is the subset of free monomial system S^* generated by set $\{x,y\}$. Suppose T^* is the free monomial system created by T, the set of all one-respect lattice paths starting from point (0,0) and ending at the straight line $y=x$. So, we have

$$\sum_{u\in T^*}u=\left(1-\sum_{u\in T}u\right)^{-1}. \quad (A.2.9)$$

Because the total number of lattice paths from (0,0) to (n,n) is C_{2n}^{n}, we have

$$\sum_{u\in T^*}u=\sum_{n=0}^{\infty}C_{2n}^{n}(xy)^n. \quad (A.2.10)$$

Apparently, sum $\Sigma_{a\in T}a$ determines mother function $f(x,y)$ of the one-respect lattice paths. Now we have

$$\sum_{n=0}^{\infty}C_{2n}^{n}(xy)^n=[1-f(x,y)]^{-1}. \quad (A.2.11)$$

Using formula (A.2.6), we can obtain the result that the mother function of the lattice paths in set T^* is $(1-4xy)^{1/2}$, and mother function $f(x,y)$ of the one-respect lattice paths is

$$f(x,y)=1-(1-4xy)^{1/2}=2\frac{1-\sqrt{1-4xy}}{2}. \quad (A.2.12)$$

As we have noticed, there are two types of one-respect lattice paths. The lattice paths in one type are always over straight line $y=x$, while the lattice paths in the other are under straight line $y=x$. These two types of one-respect lattice paths are symmetric with respect to straight line $y=x$, so the number of one-respect lattice paths with length $2n$ in each type is $b_n/2$ and the corresponding mother function is $f(x,y)/2$.

System R. Let us consider a lattice path starting from point (0,0) and arriving at the straight line $y=x-1$ never surpassing this line. These lattice paths form set R. All one-direction paths are formed by linking several lattice paths of set R. The mother function of the lattice paths for set R is

$$\frac{1-\sqrt{1-4xy}}{2y}. \quad (A.2.13)$$

System about one-direction lattice paths R^*. The free monomial system R^* is formed by all one-direction lattice paths which are under straight line $y=x$ rigorously, except the starting point. As mentioned above, their essential elements are lattice paths in set R.

It can be proved that the total number of lattice paths from (0,0) to (m,n) in R^* is $[(m-n)/(m+n)]_{m+n}^{n}$, $m\geqslant n$. The mother function of the lattice paths in R^* is

$$\sum_{m\geqslant n}\frac{m-n}{m+n}C_{m+n}^{n}x^my^n=\sum_{l=0}^{\infty}\left(\frac{1-\sqrt{1-4xy}}{2y}\right)^l=\left(1-\frac{1-\sqrt{1-4xy}}{2y}\right)^{-1}. \quad (A.2.14)$$

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